Winding number and non-BPS bound states of walls in nonlinear sigma models

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Nonsupersymmetric multiwall configurations are generically unstable. It is proposed that stabilization in compact space can be achieved by introducing a winding number into the model. A Bogomol'nyi-Prasad-Sommerfield- (BPS)-like bound is studied for the energy of a configuration with nonvanishing winding number. The winding number is implemented in an $\mathcal{N}=1$ supersymmetric nonlinear sigma model with two chiral scalar fields, and bound states of BPS and anti-BPS walls are found to exist in noncompact spaces. Even in the compactified space S^1, this nontrivial bound state persists above a critical radius of the compact dimension.

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I. INTRODUCTION

Extended objects such as walls have attracted much attention recently, mainly because of the possibility of a "brane world" scenario where our four-dimensional spacetime is realized on a wall embedded in a higher dimensional spacetime [1,2]. Supersymmetry (SUSY) is one of the most promising ideas to solve the hierarchy problem in unified theories [3]. Walls preserving half of the original SUSY [4–6] are called 1/2 Bogomol'nyi-Prasad-Sommerfield (BPS) states [7]. Junctions preserving 1/4 of the SUSY have also been constructed [8]. An interesting model with a two chiral scalar fields has also been found allowing BPS two-wall configurations [9] whose properties are studied with a certain ansatz [10]. By combining the brane-world scenario with SUSY, we have previously proposed a simple mechanism of SUSY breaking due to the coexistence of BPS and anti-BPS walls [11]. We have also invented another model which allows a non-BPS configuration to be absolutely stable because of the winding number [12].

Motivated by SUSY field theories in spacetime with dimensions higher than four [13,14], walls and junctions have been studied in nonlinear sigma models with four supercharges [15]. In order to preserve $\mathcal{N}=1$ SUSY in four dimensions, only holomorphic field redefinitions are allowed. With the holomorphic field redefinitions, one can transform SUSY field theories with minimal kinetic terms (linear sigma models) into those with nonminimal kinetic terms (nonlinear sigma models). The transformations give a model equivalent to the original model as far as local properties in target space are relevant. However, new physical effects can arise if the global properties in the target space are different. A typical global property of that kind is the winding number in target space [12,15]. BPS walls in compactified base space have been considered, especially in the context of two-dimensional SUSY theories, and the importance of winding number has also been noticed previously [16,17].

A typical model admitting winding number is the sine-Gordon model

$$\mathcal{L}=-\frac{1}{2}(\partial_\mu\psi)^2-\frac{1}{2}\cos^2\psi. \tag{1.1}$$

Because of the periodic field variable $\psi=\psi+2\pi$, the topology of the field space is S^1. When the base space is compactified ($y=y+2\pi R$) to $S^1\times\mathbf{R}^3$, we obtain the winding number $\pi_1(S^1)$ of the mapping $y\rightarrow\psi$. If we rewrite the same sine-Gordon model in terms of the nonperiodic variable $\phi=\sin\psi$,

$$\mathcal{L}=-\frac{1}{2}\frac{(\partial_\mu\phi)^2}{1-\phi^2}-\frac{1}{2}(1-\phi^2), \tag{1.2}$$

it is difficult to recognize the winding number, although the model should be the same as long as global properties like the winding number are irrelevant. For instance, the spectrum of fluctuations is identical for the zero winding number sector. A similar phenomenon occurs in the case of SUSY field theories. If we supersymmetrize the sine-Gordon model in four dimensions, we need a complex scalar field $\psi=\psi_R+i\psi_I$. The bosonic part of the Lagrangian reads

$$\begin{aligned}\mathcal{L}&=-\partial_\mu\psi^*\partial^\mu\psi-|\cos\psi|^2\\&=-(\partial_\mu\psi_R)^2-(\partial_\mu\psi_I)^2-(\cos\psi_R\cosh\psi_I)^2\\&\quad-(\sin\psi_R\sinh\psi_I)^2.\end{aligned} \tag{1.3}$$

Since the real part is a periodic variable $\psi_R=\psi_R+2\pi$, the topology of the field space is now naturally identified as $S^1\times\mathbf{R}$. Therefore we can define the winding number $\pi_1(S^1)$ of the mapping $y\rightarrow\psi_R$ from the compact base space $y=y+2\pi R$. We have previously found an exact non-BPS solution of two walls whose stability is guaranteed by the winding number for this model [12].

The purpose of this paper is to propose a general method to construct non-BPS configurations by introducing the winding number and to study the properties of such non-BPS wall configurations, especially the possible non-BPS bound states of walls. We introduce the winding number by constructing a nonlinear sigma model. This can be achieved by a holomorphic field redefinition transforming the field variable into an angular variable winding around a target space of nontrivial topology such as S^1. We obtain a non-BPS configuration consisting of a BPS and an anti-BPS configuration

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by giving half the winding number to each (anti-)BPS configuration. For models with a single chiral scalar field with only real parameters, we can establish a BPS-like bound: configurations with nonvanishing winding number consisting of n (anti-)BPS states have energies larger than or equal to the sum of energies of these n (anti-)BPS states. Since a superposition of these n (anti-)BPS states becomes a solution when these BPS states are far apart, our bound implies that no stable bound state can be formed in this class of model with a single chiral scalar field.

Although we do not find exact solutions with nonvanishing winding number, we can still construct an ansatz of a non-BPS configuration, which is a superposition of BPS and anti-BPS solutions in terms of the periodic variable, to give a nonvanishing winding number. This ansatz is tested in a model with a single chiral scalar field and gives a repulsion between BPS and anti-BPS walls and produces no bound state in accordance with our BPS-like bound. In contrast, a similar ansatz for configurations without winding number gives an attraction and shows annihilation into the ordinary vacuum.

The model with two chiral scalar fields admits BPS two-wall configurations with a moduli parameter corresponding to the separation of two walls within the BPS state [9]. This internal structure of the BPS state offers a new possibility of forming a bound state of BPS and anti-BPS states, whose stability is guaranteed by the nonvanishing winding number. We construct an ansatz of four walls comprising two BPS walls and another two anti-BPS walls by superposing these solutions in terms of the periodic variable. We find that the BPS-like bound allows the possibility of configurations whose energy is lower than the sum of BPS and anti-BPS states. We evaluate the energy density of the configuration as a function of the moduli and of the distance between BPS and anti-BPS states. We find an interesting nontrivial behavior of the energy density. We first study configurations in noncompact space in order to find a bound state of BPS and anti-BPS states. For one choice of intermediate vacuum, we find an absolute minimum of energy which is lower than the sum of the BPS and anti-BPS states. Although we use a variational ansatz which is guaranteed to be a solution only in the limit of infinite separation, the mere existence of the configurations whose energies are lower than the sum of the BPS and anti-BPS states is sufficient to conclude that the bound state exists. The distance between BPS and anti-BPS states and the moduli of these states are approximately evaluated using our ansatz. For another choice of intermediate vacuum, we find a local minimum at vanishing separation between BPS and anti-BPS states. This suggests an unstable bound state at the coincident limit of BPS and anti-BPS states. For compact space, we always find a minimum of energy when the BPS and anti-BPS states are equally spaced. This is due to a tendency to repel each other, as indicated by the BPS-like bound. For the same reason, we can expect that the bound state that we find in the other choice of intermediate vacuum may disappear when the radius of the compact dimension is too small. In fact, we find that the absolute minimum is gradually raised as the radius decreases, and disappears below a critical radius.

In the next section, a method is given to introduce the winding number by a holomorphic field redefinition, and a BPS-like bound is derived for models with a single chiral scalar field. In Sec. III, the winding number is introduced into a model with two chiral scalar fields. The energy of non-BPS multiwall configurations is studied numerically and a bound state of BPS and anti-BPS states is obtained.

II. WINDING NUMBER IN SUSY NONLINEAR SIGMA MODELS

A. Introducing the winding number

In order to illustrate our ideas in a simple context, we consider three-dimensional domain walls in four-dimensional field theories with four supercharges. A general Wess-Zumino model with an arbitrary number of chiral superfields Φ^i, a superpotential $\mathcal{W}$, and a Kähler potential $K(\Phi^i,\Phi^{*j})$ is given by

$$\mathcal{L}=K(\Phi^i,\Phi^{*j})\big|_{\theta^2\bar{\theta}^2}+[\mathcal{W}(\Phi^i)\big|_{\theta^2}+\text{H.c.}]. \tag{2.1}$$

We shall denote the scalar component of the superfield $\Phi^i(x,\theta,\bar{\theta})$ as $\phi^i(x)$. Let us suppose that we have a wall configuration which depends only on $x^2=y$. If the following BPS equation is satisfied, two out of the four supercharges are conserved [4–6]:

$$\frac{\partial\phi^i}{\partial y}=K^{ij^*}\frac{\partial\mathcal{W}^*(\phi^*)}{\partial\phi^{*j}}. \tag{2.2}$$

We call such a configuration a BPS wall. The other two supercharges are conserved if the similar equation with opposite sign is satisfied:

$$\frac{\partial\phi^i}{\partial y}=-K^{ij^*}\frac{\partial\mathcal{W}^*(\phi^*)}{\partial\phi^{*j}}. \tag{2.3}$$

We call such a configuration an anti-BPS wall. Since these walls connect two supersymmetric vacua, we need models with two vacua at least. The simplest model has a single chiral scalar field Φ with a minimal kinetic term and a cubic superpotential $\mathcal{W}$,

$$\begin{aligned}\mathcal{L}&=\Phi^\dagger\Phi\big|_{\theta^2\bar{\theta}^2}+\left[\left(\frac{m^2}{g}\Phi-\frac{g}{3}\Phi^3\right)\bigg|_{\theta^2}+\text{H.c.}\right]\\&=-\frac{\partial\phi}{\partial x_m}\frac{\partial\phi^*}{\partial x^m}-\left|\frac{m^2}{g}-g\phi^2\right|^2+\text{fermions}.\end{aligned} \tag{2.4}$$

The BPS Eq. (2.2) and anti-BPS Eq. (2.3) have the solutions

$$\phi_{(\text{wall})}(y)=\frac{m}{g}\tanh[m(y-y_0)], \tag{2.5}$$

$$\phi_{(\text{antiwall})}(y)=-\frac{m}{g}\tanh[m(y-\bar{y}_0)], \tag{2.6}$$

representing walls located at y_0 and $\overline{y_0}$, respectively. For a compact space $y = y + 2\pi R$, we have also found an exact solution of the wall and antiwall configuration which breaks supersymmetry completely [11]:

$$\phi_{(\text{wall-antiwall})}(y) = \frac{m}{g}\frac{k\sqrt{2}}{\sqrt{1+k^2}}\text{sn}\left(\frac{\sqrt{2}}{\sqrt{1+k^2}}my,k\right), \quad (2.7)$$

where $\text{sn}(u,k)$ is the Jacobi elliptic function, $0 \leqslant k \leqslant 1$, and $R = \sqrt{2}\sqrt{1+k^2}K(k)/(\pi m)$, where $K(k)$ is the complete elliptic integral. This non-BPS solution corresponds to a wall located at $y=0$ and an antiwall at $y=\pi R$. The small fluctuations around this background exhibit a tachyon corresponding to wall-antiwall annihilation instability [11].

A promising idea for stabilizing the non-BPS configuration of two walls is to introduce a topological quantum number, typically a winding number, into the model. We give a topology of S^1 to field space so that we can have a notion of winding from a compactified base space which is also S^1. To achieve that goal, we make a holomorphic redefinition of the field ϕ into a periodic one ψ:

$$\phi(x) = \frac{m}{g}\sin\psi(x),$$

$$\Phi(x,\theta,\bar{\theta}) = \frac{m}{g}\sin\Psi(x,\theta,\bar{\theta}). \quad (2.8)$$

In terms of the periodic variable ψ, the model (2.4) becomes

$$\mathcal{L} = \frac{m^2}{g^2}\sin\Psi^\dagger\sin\Psi\Big|_{\theta^2\bar{\theta}^2} + \frac{m^3}{g^2}\left(\sin\Psi - \frac{1}{3}\sin^3\Psi\right)\Big|_{\theta^2} + \text{H.c.}$$
$$= -\frac{m^2}{g^2}\left|\cos\psi\frac{\partial\psi}{\partial x^m}\right|^2 - \left|\frac{m^2}{g}\cos^2\psi\right|^2 + \text{fermions}. \quad (2.9)$$

The field space now acquires the topology of $S^1 \times \mathbf{R}$. Then the SUSY vacua occur at $\psi = \pi(n+1/2)$ with the periodicity $\psi = \psi + 2\pi$. The BPS equation (2.2) becomes

$$\frac{d\psi}{dy} = \cos\psi. \quad (2.10)$$

The BPS solution (2.5) is mapped into a solution of this transformed BPS Eq. (2.10):

$$\psi_{(\text{BPS})}(y) = \arcsin\{\tanh[m(y-y_0)]\} \quad (2.11)$$

connecting the SUSY vacuum $\psi = -\pi/2$ at $y = -\infty$ to $\psi = \pi/2$ at $y = \infty$. The solution of the anti-BPS equation connecting the SUSY vacuum $\psi = \pi/2$ at $y = -\infty$ to $\psi = 3\pi/2$ at $y = \infty$ is given by

$$\psi_{(\text{antiBPS})}(y) = \arcsin\{\tanh[m(y-\overline{y_0})]\} + \pi. \quad (2.12)$$

Since the value of the field ψ at the right end of the BPS wall (2.11) is the same as at the left end of the anti-BPS wall (2.12), there is a possibility of connecting these two wall solutions located at $y_0 < \overline{y_0}$. Such a field configuration should have winding number 1.

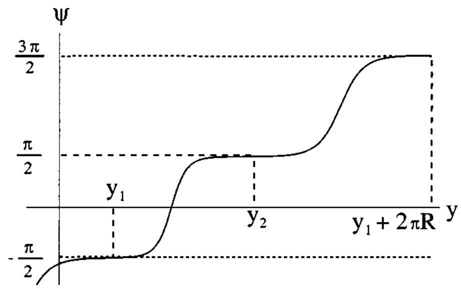

FIG. 1. The profile of the field configuration of ψ with unit winding number. The dotted lines $\psi = -\pi/2$ and $\psi = 3\pi/2$ are identified.

In fact we have found previously that a similar model with the minimal kinetic term provides the same BPS equation (2.10) and that there is an exact solution for the non-BPS configuration of two walls for compactified space y [12]. The configuration was found to wind around the field space ψ once and is topologically stable.[1]

In our model with the periodic variable (2.9) we have an exact solution for compactified space:

$$\psi_{(\text{wall-antiwall})}(y) = \arcsin\left[\frac{m}{g}\frac{k\sqrt{2}}{\sqrt{1+k^2}}\text{sn}\left(\frac{\sqrt{2}}{\sqrt{1+k^2}}my,k\right)\right], \quad (2.13)$$

obtained by transforming the non-BPS solution (2.7) to our periodic variable ψ. Since $0 < k\sqrt{2}/\sqrt{1+k^2} < 1$ for $0 < k < 1$, the configuration has no winding number and represents the wall-antiwall configuration as in our original model without periodic variable [11]. One also finds that the small fluctuation around the background has exactly the same spectrum, including the tachyon instability. This is consistent with the fact that the ordinary vacuum is the minimum energy configuration in the vanishing winding number sector.

B. BPS-like bound for winding field configuration

We are interested in the field configuration with a nonvanishing winding number. Let us consider a BPS-like bound for the energy of the configuration with a nonvanishing winding number. Let us first consider the model (2.9) as the simplest model for illustration. If there is a field configuration with unit winding number, ψ should rotate by 2π as y increases by $2\pi R$. The noncompact space is obtained by the limit $R \to \infty$. Let us call the point y_1 where $\psi = -\pi/2$ and y_2 where $\psi = \pi/2$ and assume $y_1 < y_2$ as illustrated in Fig. 1. The energy of a configuration with one winding number is given in one periodicity domain by

[1] Since the 1/2 BPS solution is real, we can reinterpret the inverse of the Kähler metric K^{ij*} as a part of the derivative of a superpotential $K^{ij*}\partial\mathcal{W}/\partial\phi^{*j} = \partial\tilde{\mathcal{W}}/\partial\phi^{*j}$, as noted in [15].

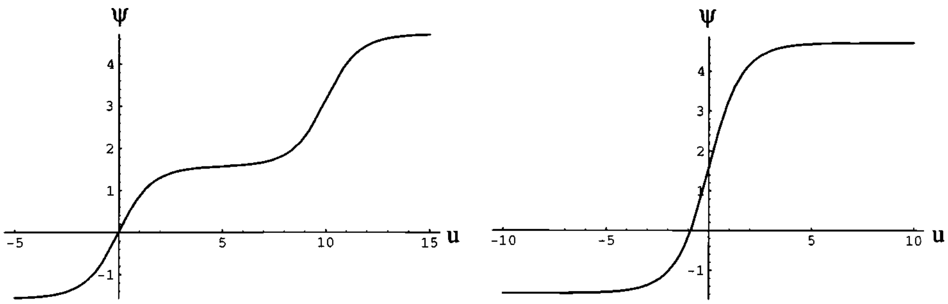

FIG. 2. The profile of ψ with unit winding number; the superposition of the BPS wall and the anti-BPS wall at $ma=10$ (a) and $ma=0$ (b).

$$
\begin{aligned}
E=&\int_{y_1}^{y_1+2\pi R} dy\left[\frac{m^2}{g^2}\left|\cos\psi\frac{\partial\psi}{\partial y}\right|^2+\left|\frac{m^2}{g}\cos^2\psi\right|^2\right]\\
=&\int_{y_1}^{y_2}dy\left[\left|\frac{m}{g}\cos\psi\frac{\partial\psi}{\partial y}-\frac{m^2}{g}\cos^2\psi\right|^2+\frac{m^3}{g^2}\frac{d}{dy}\left(\sin\psi\right.\right.\\
&\left.\left.-\frac{1}{3}\sin^3\psi\right)\right]+\int_{y_2}^{y_1+2\pi R}dy\left[\left|\frac{m}{g}\cos\psi\frac{\partial\psi}{\partial y}+\frac{m^2}{g}\cos^2\psi\right|^2\right.\\
&\left.-\frac{m^3}{g^2}\frac{d}{dy}\left(\sin\psi-\frac{1}{3}\sin^3\psi\right)\right]\\
\geqslant&\left[\frac{m^3}{g^2}\left(\sin\psi-\frac{1}{3}\sin^3\psi\right)\right]_{y_1}^{y_2}\\
&-\left[\frac{m^3}{g^2}\left(\sin\psi-\frac{1}{3}\sin^3\psi\right)\right]_{y_2}^{y_1+2\pi R}=2E_{\mathrm{BPS}},
\end{aligned}
\tag{2.14}
$$

where E_{BPS} is the energy of the single BPS or anti-BPS wall. Therefore any configuration with unit winding number has energy larger than or equal to the sum of the energies of a BPS wall and an anti-BPS wall. Since this superposition of the BPS and anti-BPS states becomes a solution of the equation of motion as their separation goes to infinity, we find that the BPS state and the anti-BPS state in the unit winding number sector always repel each other when they are sufficiently far apart at least. Whether there is any local minimum for finite separation or not is the remaining question which we will address in the next subsection.

This BPS-like bound can also be generalized to other models of a single chiral scalar field using an arbitrary superpotential with real parameters. This may be achieved if the parameters of the model can be made real by phase rotations of the fields. Then we can assume that the field configuration is real. Let us suppose that there are two vacua at ψ_1 and ψ_2 of the periodic variable $\psi=\psi+2\pi$. Without loss of generality we can assume $\mathcal{W}(\psi_1)<\mathcal{W}(\psi_2)$. If there is a field configuration with a single winding number which takes the value ψ_1 at y_1 and ψ_2 at y_2, we can apply a BPS bound for the interval $y_1<y<y_2$ and an anti-BPS bound for the interval $y_2<y<y_1+2\pi R$ as in Eq. (2.14). Thus we obtain that the energy of the field configuration with a single winding number is bounded by $2E_{\mathrm{BPS}}=2[\mathcal{W}(\psi_2)-\mathcal{W}(\psi_1)]$. Similarly, we can easily find that winding field configurations consisting of n (anti-)BPS states in one periodicity domain $0\leqslant y\leqslant 2\pi R$ have energy larger than or equal to the sum of these BPS states.

C. Repulsion between BPS and anti-BPS walls

Since we cannot find exact solutions in the sector with nonvanishing winding number, we shall use an approximate evaluation of the possible configurations inspired by the BPS (2.11) and anti-BPS (2.12) solutions. This is at least sufficient to give an upper bound of the energy of the possible minimum energy solution from the viewpoint of a variational approach.

Let us consider a superposition of the BPS wall solution (2.11) centered at $y=0$ and the anti-BPS solution (2.12) centered at $y=a$,

$$
\psi(y)=\arcsin[\tanh(my)]+\arcsin\{\tanh[m(y-a)]\}+\frac{\pi}{2}
\tag{2.15}
$$

connecting the SUSY vacuum $\psi=-\pi/2$ at $y=-\infty$ to $\psi=3\pi/2$ at $y=\infty$, and having unit winding number. Although this is not a static solution of the equation of motion for finite separation a, it reduces to a solution in the limit $a\to\infty$. Defining a dimensionless coordinate

$$
u\equiv my,\quad u_a\equiv ma,
\tag{2.16}
$$

the energy of this configuration is found to be

$$
\begin{aligned}
E=&\int_{-\infty}^{\infty}dy\left[\frac{m^2}{g^2}\left|\cos\psi\frac{\partial\psi}{\partial y}\right|^2+\left|\frac{m^2}{g}\cos^2\psi\right|^2\right]\\
=&\frac{m^3}{g^2}\int_{-\infty}^{\infty}du\left[\left(\frac{\tanh u}{\cosh(u-u_a)}+\frac{\tanh(u-u_a)}{\cosh u}\right)^2\right.\\
&\times\left(\frac{1}{\cosh u}+\frac{1}{\cosh(u-u_a)}\right)^2\\
&\left.+\left(\frac{\tanh u}{\cosh(u-u_a)}+\frac{\tanh(u-u_a)}{\cosh u}\right)^4\right].
\end{aligned}
\tag{2.17}
$$

One should note that the field (2.15) as well as its derivative are nonsingular in the entire region of y. On the other hand, the energy density in the integrand of Eq. (2.17) has contributions from the kinetic term (first term) and the potential term (second term), both of which have powers of $\cos\psi$

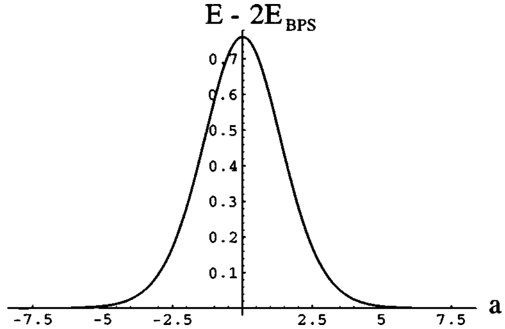

FIG. 3. The energy of ψ as a function of the wall separation a $(m=1,g=1)$.

vanishing at vacua. As a consequence, the energy density of the two-wall configuration in Eq. (2.17) has a zero at $y=a/2$ for any values of $a\geqslant 0$ and exhibits two separated peaks for two walls. This is true even for $a=0$ where the field ψ itself shows no sharp separation of two walls as shown in Fig. 2(b).

A typical field configuration at $u_a=ma=10$ in Fig. 2a shows ψ winding once to form two walls. Even at the coincident limit $a\to 0$ of two walls, the field configuration is nontrivial as shown in Fig. 2b. The energy as a function of the wall separation a is shown in Fig. 3, where the parameters are set to $m=1,g=1$. We see that the energy is always larger than the sum of the isolated wall and antiwall and reduces to the sum at $a\to\infty$ in accordance with our BPS-like bound. Therefore we find that a BPS wall and an anti-BPS wall repel each other and have no stable bound state in the unit winding number sector.

To examine how well our ansatz carries the correct behavior of the lowest energy configuration, we also compute the energy of the corresponding ansatz in the vanishing winding number sector:

$$\psi_{\text{no winding}}(y)=\arcsin[\tanh(my)]-\arcsin\{\tanh[m(y-a)]\}-\frac{\pi}{2}, \tag{2.18}$$

$$E_{\text{no winding}}=\frac{m^3}{g^2}\int_{-\infty}^{\infty}du\left[\left(\frac{\tanh u}{\cosh(u-u_a)}-\frac{\tanh(u-u_a)}{\cosh u}\right)^2\left(\frac{1}{\cosh u}-\frac{1}{\cosh(u-u_a)}\right)^2+\left(\frac{\tanh u}{\cosh(u-u_a)}-\frac{\tanh(u-u_a)}{\cosh u}\right)^4\right]. \tag{2.19}$$

A typical field configuration at $u_a=10$ in Fig. 4a shows no winding to be compared with Fig. 2a. At the coincident limit $a\to 0$ of two walls, the field ψ becomes constant as shown in Fig. 4b and reduces to the ordinary vacuum $\psi=-\pi/2$ in contrast to the unit winding number case in Fig. 2b. In Fig. 5 we show the energy of the two-wall configuration in the vanishing winding number sector as a function of the wall separation a. It reduces to the sum of the BPS energies of two walls at $a\to\pm\infty$ and vanishes at the coincident point $a=0$. This clearly shows that the wall-antiwall configuration in the zero winding number sector is unstable and annihilates into the vacuum.

III. WINDING NUMBER IN A MODEL WITH TWO FIELDS

A. Model with two fields

In order to explore the nontrivial behavior of the winding number configuration, we consider the next simplest possibility, models with two chiral scalar fields. It has been found that the model with minimal kinetic terms for chiral scalar fields Φ and X with the following superpotential $\mathcal{W}$ has an integral of motion [9]:

$$\mathcal{W}(\Phi,X)=\frac{m^2}{g}\Phi-\frac{g}{3}\Phi^3-\frac{g}{4}\Phi X^2,\quad m,g>0. \tag{3.1}$$

This model is the simplest modification of our model in Eq. (2.4) in the previous section to allow four degenerate SUSY vacua at $(\phi,\chi)=(\pm m/g,0),(0,\pm 2m/g)$. There are BPS wall solutions connecting the vacuum $(-m/g,0)$ to $(0,\pm 2m/g)$ and solutions $(0,\pm 2m/g)$ to $(m/g,0)$. Since both of them turn out to conserve the same supercharge, the smooth connection of these wall solutions located far apart should be a solution of the same BPS equation. A remarkable property of this model is that it admits a BPS solution of two walls connecting $(\phi,\chi)=(-m/g,0)$ to $(m/g,0)$:

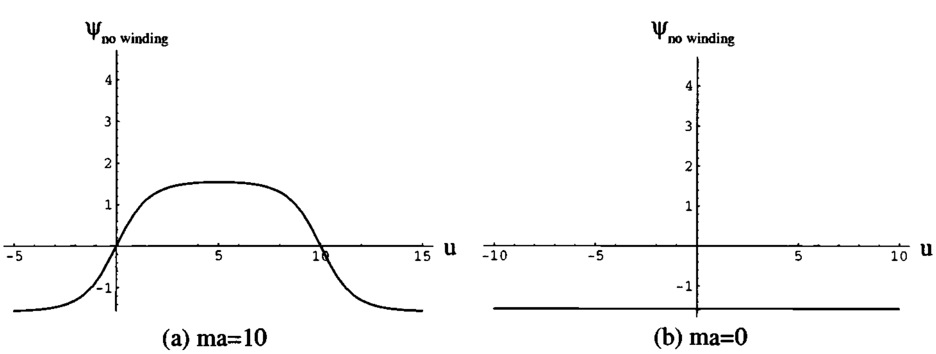

FIG. 4. The profile of $\psi_{\text{no winding}}$ without winding number; the superposition of the BPS wall and the anti-BPS wall at $ma=10$ (a) and $ma=0$ (b).

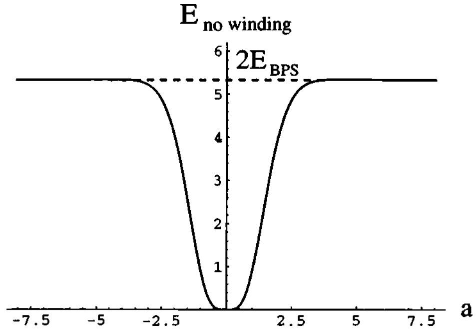

FIG. 5. The energy of $\psi_{\text{no winding}}$ as a function of the wall separation a $(m=1,g=1)$.

$$\phi=\frac{m}{g}f(u), \quad f(u)=\frac{\sinh u}{\cosh u+t},$$

$$\chi=\pm\frac{m}{g}h(u), \quad h(u)=2\sqrt{\frac{t}{\cosh u+t}}, \tag{3.2}$$

where $0\leq t$ is a moduli parameter [9,10]. This configuration can be interpreted as a smooth connection of the above two BPS walls connecting between $(-m/g,0)$ and $(0,\pm 2m/g)$ and between $(0,\pm 2m/g)$ to $(m/g,0)$. They are centered at $y=0$. If the moduli parameter t is larger than 1, these two walls are separated by a distance $y=\hat{t}$ where $\cosh(m\hat{t})=t$. The case with $0\leq t<1$ corresponds to two walls compressed into each other so that the walls merge together completely.

We introduce the concept of winding number by making a holomorphic change of variable (2.8) from ϕ to a periodic one $\psi=\arcsin(g\phi/m)$. The bosonic part of our model with the periodic variable is then given by

$$\mathcal{L}_{\text{bosonic}}=-\frac{m^2}{g^2}\left|\cos\psi\frac{\partial\psi}{\partial x^m}\right|^2-\left|\frac{\partial\chi}{\partial x^m}\right|^2-\left|\frac{m^2}{g}\cos^2\psi-\frac{g}{4}\chi^2\right|^2 -\left|-\frac{m}{2}\chi\sin\psi\right|^2. \tag{3.3}$$

The vacua of this model are at

$$(\psi,\chi)=\left(\frac{\pi}{2}+n\pi,0\right), \quad \left(n\pi,\frac{2m}{g}\right), \quad \left(n\pi,-\frac{2m}{g}\right), \tag{3.4}$$

with integer $-\infty<n<\infty$. The BPS equations are given by

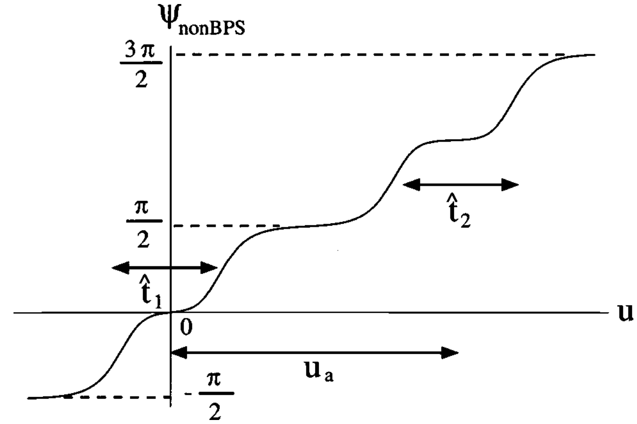

FIG. 6. The profile of the field configuration of $\psi_{\text{non-BPS}}$.

$$\cos\psi\frac{\partial\psi}{\partial u}=\cos^2\psi^*-\frac{g^2}{4m^2}\chi^{*2},$$

$$\frac{\partial\chi}{\partial u}=-\frac{1}{2}\sin\psi^*\chi^*, \quad u\equiv my. \tag{3.5}$$

One can obtain from Eq. (3.2) a BPS two-wall solution which connects $(\psi,\chi)=(-\pi/2,0)$ at $y=-\infty$ to $(\psi,\chi)=(\pi/2,0)$ at $y=\infty$:

$$\psi_{\text{BPS}}=\arcsin[f(u)], \quad \chi^{\pm}_{\text{BPS}}=\pm\frac{m}{g}h(u) \tag{3.6}$$

where the functions $f(u),h(u)$ are defined in Eq. (3.2). The $\pm$ of $\chi^{\pm}$ corresponds to the sign of the vacuum $(\psi,\chi)=(0,\pm 2m/g)$ in the intermediate region. Another two-wall configuration connecting $(\psi,\chi)=(\pi/2,0)$ at $y=-\infty$ to $(\psi,\chi)=(3\pi/2,0)$ at $y=\infty$ is given as a solution of the anti-BPS equation preserving the opposite combination of supercharges:

$$\psi_{\text{anti-BPS}}=\arcsin[f(u)]+\pi, \quad \chi^{\pm}_{\text{anti-BPS}}=\pm\frac{m}{g}h(u). \tag{3.7}$$

These solutions are centered at $y=0$ and have a moduli t.

We can construct an ansatz for the configuration with unit winding number by superposing the BPS two-wall solution with moduli t_1 centered at $y=0$ and the anti-BPS two-wall solution with the moduli t_2 centered at $y=a$:

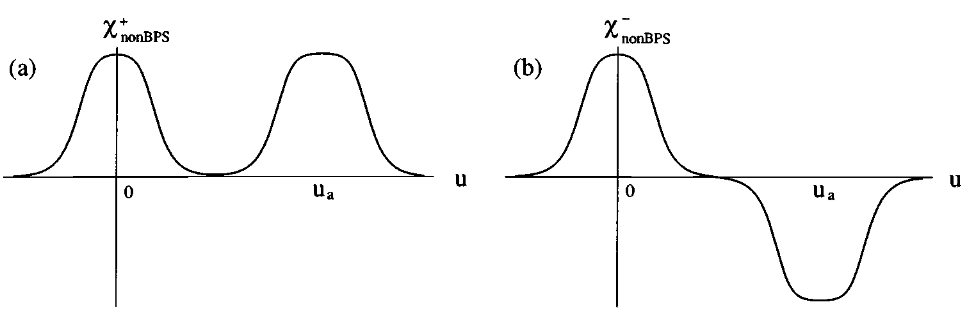

FIG. 7. The profile of the field configuration of $\chi^{\pm}_{\text{non-BPS}}$.

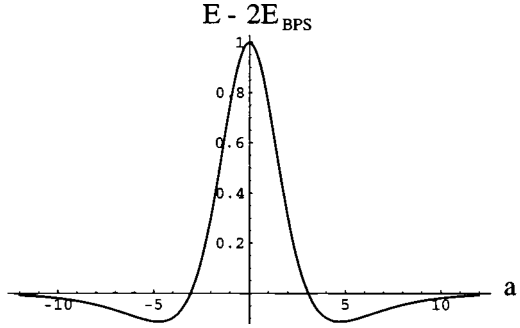

FIG. 8. The energy of the field configuration of (ψ,χ^+) as a function of a at $t=0.2$ $(m=1,g=1)$.

$$\psi_{\text{non-BPS}}=\arcsin\left(\frac{\sinh u}{\cosh u+t_1}\right)+\arcsin\left(\frac{\sinh(u-u_a)}{\cosh(u-u_a)+t_2}\right)+\frac{\pi}{2},$$

$$\chi^{\pm}_{\text{non-BPS}}=\frac{2m}{g}\left(\sqrt{\frac{t_1}{\cosh u+t_1}}\pm\sqrt{\frac{t_2}{\cosh(u-u_a)+t_2}}\right), \tag{3.8}$$

where the sign $\pm$ for the field $\chi^{\pm}$ indicates the same $(+)$ or opposite $(-)$ sign of the vacuum in the intermediate region around $y=0$ and around $y=a$. A typical field configuration (ψ,χ^+) is illustrated in Figs. 6 and 7a. We have shown the distance $\hat{t}_i\equiv\text{arccosh}(t_i)/m$ between walls within the BPS state $(i=1)$ and the anti-BPS state $(i=2)$. For comparison, the field configuration (ψ,χ^-) for the vacuum $(\psi,\chi)=(0,-2m/g)$ in the intermediate region is illustrated in Figs. 6 and 7b.

B. BPS-like bound for two-field model

Let us first examine what sort of BPS-like bound can be obtained in the case of two fields. The energy in the one-periodicity domain is given by

$$E=\int_0^{2\pi R}dy\left[\left|\frac{m}{g}\cos\psi\frac{d\psi}{dy}\right|^2+\left|\frac{d\chi}{dy}\right|^2+\left|\frac{m^2}{g}\cos^2\psi-\frac{g}{4}\chi^2\right|^2+\left|-\frac{m}{2}\chi\sin\psi\right|^2\right]. \tag{3.9}$$

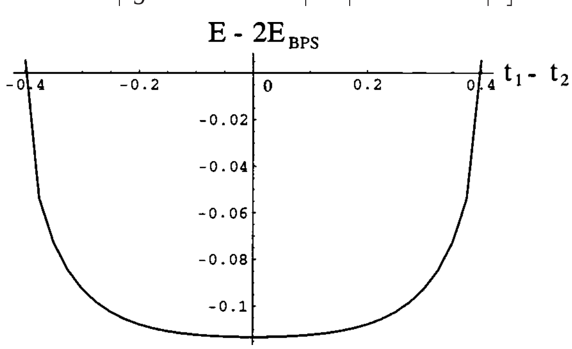

FIG. 9. The energy of (ψ,χ^+) as a function of t_1-t_2 at $(t_1+t_2)/2=0.2$ and $ma=4.73$ $(m=1,g=1)$.

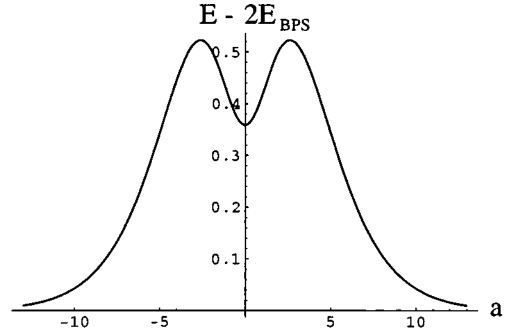

FIG. 10. The energy of the field configuration of (ψ,χ^-) as a function of a at $t=0.423$ $(m=1,g=1)$.

As in the single field model, we assume that the periodic field ψ takes values $-\pi/2$ at y_1 and $\pi/2$ at y_2. We find

$$\begin{aligned}E=&\int_{y_1}^{y_2}dy\left[\left|\frac{m}{g}\cos\psi\frac{d\psi}{dy}-\frac{m^2}{g}\cos^2\psi+\frac{g}{4}\chi^2\right|^2+\left|\frac{d\chi}{dy}+\frac{m}{2}\chi\sin\psi\right|^2+\frac{d\mathcal{W}(\psi,\chi)}{dy}\right]\\&+\int_{y_2}^{y_1+2\pi R}dy\left[\left|\frac{m}{g}\cos\psi\frac{\partial\psi}{\partial y}+\frac{m^2}{g}\cos^2\psi-\frac{g}{4}\chi^2\right|^2+\left|\frac{d\chi}{dy}-\frac{m}{2}\chi\sin\psi\right|^2-\frac{d\mathcal{W}(\psi,\chi)}{dy}\right].\end{aligned} \tag{3.10}$$

In the case of the single field model, one can specify the value of the superpotential in terms of the value of the field ψ. Combined with the requirement of nonvanishing winding number and the continuity of the real function $\psi(y)$, we are sure that the superpotential has to reach the vacuum value $\mathcal{W}(\psi=\pi/2)$ before returning back to the original value $\mathcal{W}(\psi=-\pi/2)$ as dictated by the periodicity of the superpotential in our model. This is the origin of the BPS-like bound for the single field model. In the case of two fields, however, the above bound reads

$$\begin{aligned}E\geqslant&2\left[\mathcal{W}\left(\psi(y_2)=\frac{\pi}{2},\chi(y_2)\right)-\mathcal{W}\left(\psi(y_1)=\frac{-\pi}{2},\chi(y_1)\right)\right]\\=&2\left[\left(\frac{2m^3}{3g^2}-\frac{m}{4}\chi^2(y_2)\right)-\left(-\frac{2m^3}{3g^2}+\frac{m}{4}\chi^2(y_1)\right)\right]\\=&2E_{\text{BPS}}-\frac{m}{2}[\chi^2(y_2)+\chi^2(y_1)].\end{aligned} \tag{3.11}$$

TABLE I. Fractional difference $(E_5-E_3)/E_3$ between energies in compact space in approximations with five walls E_5 and three walls E_3 for $2m\pi R=10,20,40$. Difference is largest for antiwall placed at the center of periodicity $a=\pi R$, and smallest at coincident limit $a=0,2\pi R$.

$2m\pi R$	10	20	40
$a=0,2\pi R$	9.18×10^{-5}	4.12×10^{-9}	$\leqslant10^{-9}$
$a=\pi R$	1.03×10^{-2}	6.81×10^{-5}	3.09×10^{-9}

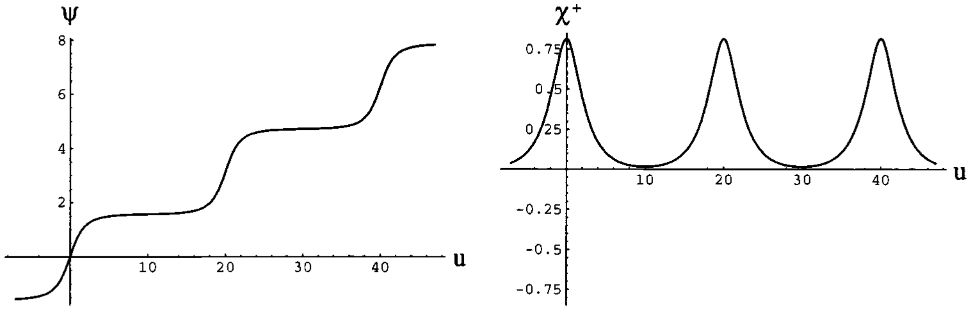

FIG. 11. The profile of the field configuration of (ψ,χ^+) in three-wall approximation at $2m\pi R=40$ and $t=0.2$ $(m=1,g=1)$.

Nonvanishing $\chi(y_1)$ and $\chi(y_2)$ make this bound lower than twice the BPS energy. In the case of two fields, it is not guaranteed for two fields to take the particular value specified by the vacuum at the same time. Consequently, the superpotential need not reach the value at the vacuum before returning back to its original value. Since we have just chosen particular points y_1, y_2 to divide the integration region in order to evaluate the BPS or anti-BPS bound, it is possible that we can have better bounds by choosing other points of division. However, it does not seem to be possible to choose such a point in general situations. Therefore we just conclude that the energy of the winding configuration need not be larger than the sum of the BPS and anti-BPS states. This result suggests that there is a possibility for a non-BPS bound state of walls in the case of the two-wall model.

C. Non-BPS multiwalls in the noncompact space

Here we present the energy (3.9) of the non-BPS configuration of four walls with unit winding number (3.8). To avoid too many parameters, we will use the same moduli parameters for both BPS and anti-BPS states, $t_1=t_2\equiv t$, except as stated otherwise.

For the case of $\chi^{\pm}$ with the same sign of the field χ for BPS and anti-BPS walls, we show the energy as a function of separation a between BPS and anti-BPS walls for fixed moduli t in Fig. 8. A typical field configuration can be obtained by letting $\hat{t}_1=\hat{t}_2$ in Figs. 6 and 7a. We observe the following.

(1) There exist configurations which have energies lower than the sum of BPS and anti-BPS states, in contrast to the single field case. Although we have only a limited ansatz of field configurations inspired by physical considerations, this fact clearly shows that the BPS and anti-BPS states infinitely far apart is not the lowest energy state in the nonvanishing winding number sector.

(2) Among these configurations, we find the minimum energy configuration with a separation $a\approx 4.73/m$ of BPS and anti-BPS states and the moduli $t\approx 0.200$ which is shown in Fig. 8. Thus we find an approximate evaluation of the minimum energy configuration for the unit winding number sector.

(3) We have also examined the energy as a function of the difference t_1-t_2 between the moduli parameters of the BPS state t_1 and the anti-BPS state t_2 as shown in Fig. 9. It shows clearly that the configuration achieves the minimum energy for $t_1=t_2$.

(4) Therefore we find that there exists a non-BPS bound state of walls whose approximate configuration is given by two BPS and anti-BPS walls with moduli parameter $t_1=t_2\approx 0.200$ which are separated by $a\approx 4.73/m$.

For the case with the opposite sign of the fields χ for BPS and anti-BPS walls, we show the energy as a function of separation a between BPS and anti-BPS walls for fixed moduli $t\approx 0.423$ in Fig. 10. A typical field configuration can be obtained by letting $\hat{t}_1=\hat{t}_2$ in Figs. 6 and 7b. We observe the following.

(1) We find that energy is always higher than the sum of the BPS and anti-BPS states.

(2) There exists a local minimum of energy at zero separation $a=0$ between BPS and anti-BPS states for the moduli parameter $t<3.705$. The lowest value of this local minimum occurs at the moduli $t\approx 0.423$ which is shown in Fig. 10.

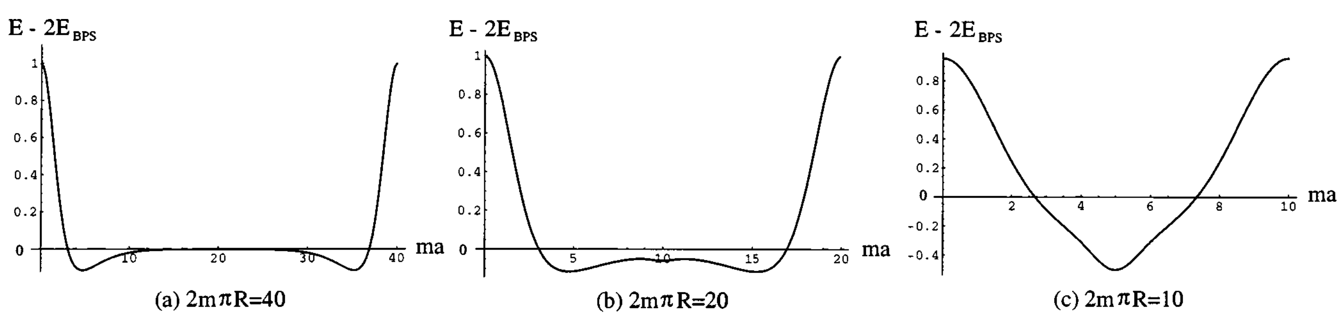

FIG. 12. The energy of (ψ,χ^+) as a function of ma at $t=0.2$ in the case of $2m\pi R=40$ (a), $2m\pi R=20$ (b), and $2m\pi R=10$ (c) $(m=1,g=1)$.

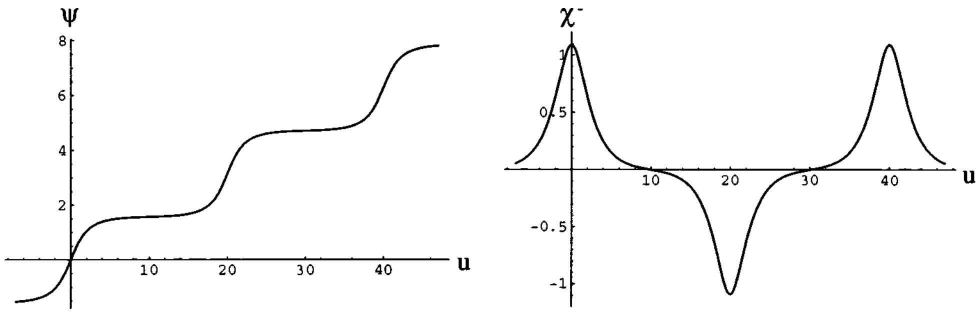

FIG. 13. The profile of the field configuration of (ψ,χ^-) in three-wall approximation at $2m\pi R=40$ and $t=0.423$.

D. Non-BPS multiwalls in compactified space

It is interesting to examine if the above non-BPS bound state of walls persists when the space is compactified on S^1, since one expects a repulsion from the other walls located at $2\pi R$, which is the periodicity of the base space $y=y+2\pi R$.

When the space is compactified, the BPS states are placed at $y=2n\pi R$ and the anti-BPS walls at $y=2n\pi R+a$ periodically. Then we have the ansatz

$$\psi_{\text{non-BPS}}=\sum_{n=-\infty}^{\infty}\{\arcsin[f(u-2nu_0)]+\arcsin[f(u-u_a-2nu_0)]+\pi\}-\frac{\pi}{2},$$

$$\chi^{\pm}_{\text{non-BPS}}=\sum_{n=-\infty}^{\infty}\left[\frac{m}{g}[h(u-2nu_0)\pm h(u-u_a-2nu_0)]\right], \tag{3.12}$$

where $u\equiv my, u_a\equiv ma$, and $u_0\equiv m\pi R$.

In the one-periodicity domain $a-\pi R<y<a+\pi R$, only nearby walls are important. Three-wall (placed at $y=0,\ a, 2\pi R$) and five-wall (placed at $y=a-2\pi R$, $0,a,2\pi R,a+2\pi R$) approximations of the energy in the interval $a-\pi R<y<a+\pi R$ are compared in the case of the moduli parameter $t=0$. We find excellent agreement between three-wall and five-wall approximations. We show the fractional difference $(E_5-E_3)/E_3$ between the energies in compact space in approximations with five walls E_5 and three walls E_3 in Table I. Therefore we choose to use the three-wall approximation in the following. The energy density in Eq. (3.9) can be evaluated in the three-wall approximation by using $f_i=f(u_i), h_i=h(u_i), i=1,2,3$, and $u_1=u, u_2=u-u_a, u_3=u-2m\pi R$:

$$\sin\psi=-f_1\sqrt{(1-f_2^2)(1-f_3^2)}-f_2\sqrt{(1-f_3^2)(1-f_1^2)}-f_3\sqrt{(1-f_1^2)(1-f_2^2)}+f_1f_2f_3,$$

$$\cos\psi=-\sqrt{(1-f_1^2)(1-f_2^2)(1-f_3^2)}+f_1f_2\sqrt{(1-f_3^2)}+f_2f_3\sqrt{(1-f_1^2)}+f_3f_1\sqrt{(1-f_2^2)},$$

$$\frac{d\psi}{du}=\sum_{i=1}^{3}\frac{1}{\sqrt{(e^{u_i}+t)(e^{-u_i}+t)}}\frac{t\cosh u_i+1}{\cosh u_i+t},\qquad i=1,2,3. \tag{3.13}$$

For the choice of the same sign of χ for BPS and anti-BPS states, a typical field configuration ψ and χ^+ is shown in Fig. 11 for $2m\pi R=40$ and $t=0.2$, choosing $a=\pi R$. The corresponding energy of the unit winding number configuration is shown in Fig. 12a as a function of ma. We find that there is an absolute minimum at $ma=4.75$, which is identical to the noncompact case. We also find a very shallow local minimum at the center $a=\pi R$. This comes about because of the compactification. The general tendency of non-BPS walls is that they exert repulsion, as we encountered in the single field case. This repulsion produces a minimum at the central position.

As we decrease the compactification radius R, the absolute minimum gets shallower as shown in Fig. 12b for the case of $2m\pi R=20$ with the same moduli $t=0.2$. Eventually the absolute minimum around $ma=4.75$ disappears and we

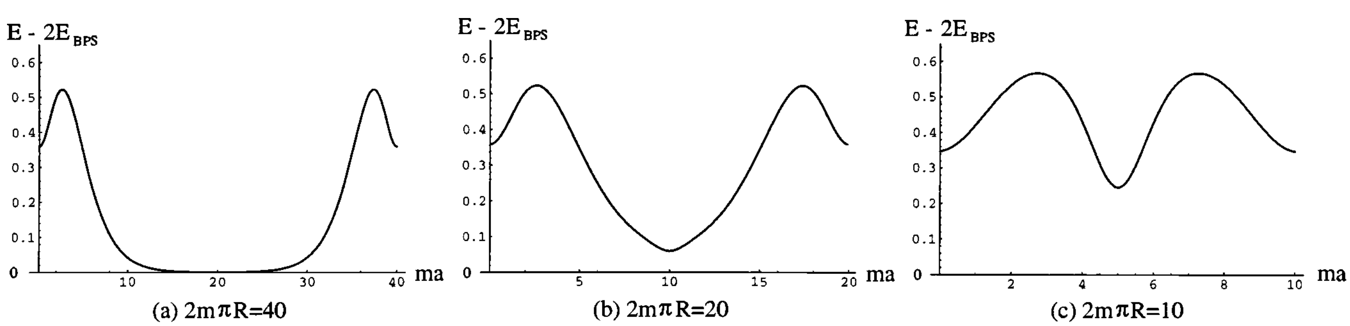

FIG. 14. The energy of (ψ,χ^-) as a function of ma at $t=0.423$ in the case of $2m\pi R=40$ (a), $2m\pi R=20$ (b), and $2m\pi R=10$ (c) $(m=1,g=1)$.

obtain only a single minimum at the center $a=\pi R$ as shown in Fig. 12c. Thus we find that the non-BPS bound state of walls persists for larger values of compactification radius up to $2m\pi R<16.92$, and that it becomes unstable for smaller values of the radius.

Let us next examine the case with the opposite sign of the fields χ for BPS and anti-BPS walls. A typical field configuration ψ and χ^- is shown in Fig. 13 for $2m\pi R=40$ and $t=0.423$, choosing $a=\pi R$. For noncompact space, we found a local minimum of energies at $a=0$. This still persists for $2m\pi R=40$, $2m\pi R=20$, and $2m\pi R=10$ as shown in Fig. 14. The absolute minimum always occurs at the center $a=\pi R$ in the case of the opposite sign of the fields χ for BPS and anti-BPS walls. Since the width of the BPS state is of order $1/m$, our ansatz requires my to span a region of a few times $1/m$ for the field ψ to make a full 2π winding. Therefore we should not trust our ansatz for too small values of radius R.

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